THE

MATHEMATICAL GAZETTE.

EDITED BY
W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF
F. S. MACAULAY, M.A., D.Sc.; PROF. H. W. LLOYD-TANNER, M.A., D.Sc., F.R.S.;
PROF. E. T. WHITTAKER, M.A., F.R.S.

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REPORT ON THE TEACHING OF MATHEMATICS IN PREPARATORY SCHOOLS.*

THE present report is issued in response to a widely expressed desire on the part of preparatory school masters. The Sub-Committee that drafted the report contained three preparatory school headmasters, two of whom were nominated by the Association of Preparatory Schools.

Before any attempt was made to draft a report, a letter was sent to over 400 preparatory schools (namely to those whose names appear in the convenient list given by the Public Schools Year Book). The letter invited suggestions as to the procedure of the Sub-Committee; the answers were collated and the suggestions received careful consideration, many of them being adopted.

In issuing the present reports, the Committee wish it to be understood that it is the education of the **average** boy that they have had under consideration. They have no desire to impose a limit upon the work of preparatory schools, but they are of opinion that the work of many boys coming on to public schools has suffered from the attempt to read higher work before the elements have been mastered. They suggest that a thorough grounding in the elementary subjects indicated in the present report should precede any progress to further studies.

The work of the average boy is tested by the Entrance examinations (as opposed to the Scholarship examinations) of the various schools. Some of these schools continue to set their own papers, and the entrance examination for the R.N. College, Osborne, is conducted by the Oxford and Cambridge Joint Board; while a great proportion of schools accept the papers of the Common Examinations Board. The report of the Committee will be submitted to these various examining bodies, and there is reason to hope that the recommendations will be accepted as a guide in future papers.

GEOMETRY.

The Geometry report published by the M.A. in 1902 was issued with the object of inducing the Universities and other examining bodies to make certain changes in their regulations. Happily the Universities have not only complied with these recommendations, but have gone further in the

*This report is published separately in pamphlet form, and may be had from the Publishers, price 3*d.* net.

path of reform. The Geometry report has therefore to some extent served its purpose.

The usual understanding at the present date as to sequence of propositions is, that "any proof of a proposition is accepted which appears to the examiners to form part of a systematic treatment of the subject."

As the result of the new schedules of public examinations, most public and preparatory schools have re-cast their geometry teaching.

The standard procedure now is to begin with a course of experimental Geometry. During this stage the pupil learns the use of mathematical instruments. But this is to be regarded only as a means towards a more important end. The use of instruments must be mastered, not for their own sake, but for this reason ; that there is no other way of experimenting, with precision, on the properties of geometrical figures.

The object, then, of the experimental stage is that the learner may gradually acquaint himself with the more striking facts of Geometry (and there is no good reason for limiting him to the geometry of two dimensions). At the same time, he will be learning to perform the simpler constructions with his instruments, the construction being learnt long before the proof of it can be understood.

All this time the wise teacher will keep himself rather in the background. He is not to give away the secret ; the boy will appreciate it much more if he discovers it for himself.

In due course the study of formal geometry will begin. But it must not be supposed that instruments are henceforth to be used for constructions only ; experimental work should still be done in leading up to new theorems.

"The formal course is to be divided into two parallel courses of (i) Theorems, (ii) Constructions."* This does not mean that the two courses must be separated in point of time ; on the contrary, the theorems and constructions should be intermingled in teaching. The meaning of the recommendation is rather this : that the truth of a theorem does not depend upon the possibility of performing, with given instruments, the necessary construction. The boy may not yet have learnt how to bisect an angle with ruler and compasses ; all the same, the angle clearly has a bisector, which could be drawn, if it is desired, by folding the paper.

As boys enter their public schools at ages varying from $12\frac{1}{2}$ to $14\frac{1}{2}$, the amount of geometry that an average boy should have covered cannot be stated as an absolutely fixed quantity. Taking the age of entry as $13\frac{1}{2}$, a boy may be expected to have attained the following standard.

(1) He should have learnt to draw neatly and accurately. It is a great convenience if boys have reached a good standard in drawing before they enter a public school. At present, public schools have to spend in teaching the use of instruments a good deal of time that they would rather use for theoretical geometry.

(2) He should be able to describe his constructions in writing.

(3) He should have learnt the nature of a logical proof ; *i.e.* he should avoid assuming in the course of the argument the truth of the proposition he seeks to prove. He should be able to state his arguments in proper form.

(4) As regards knowledge of geometrical theory, the power of solving easy riders is more valuable than that of writing out propositions. Masters are advised to give time to rider work even though it entails slow progress with propositions. The difficulty of sequence does not enter into rider work. The number of theorems, or "standard facts," that are used in solving easy riders is very small, and their order does not matter. At $13\frac{1}{2}$, a boy should be quite familiar with the facts in the following cases, and should have had much practice in applying them to riders.

* Cambridge Geometry Schedule.

(*a*) Angles at a point.
(*b*) Angles made by parallel lines and a transversal.
(*c*) The sum of the angles of a triangle, and of a polygon.
(*d*) Congruent triangles.
(*e*) The equality of the base-angles of an isosceles triangle.

He should have been well drilled in the meaning of the geometrical terms used in Book I.; many boys are quite unnecessarily hazy on these points.

A thorough grounding in the above should precede any further work in geometry.

Instruments.

The following instruments are needed:

A Ruler, graduated on one edge in inches and tenths of an inch; on the other edge in centimetres and millimetres. The edges of the ruler should be bevelled, and the graduations should extend down to the edges. The graduations should not extend to the end of the ruler, but a clear space should be left at each end. No other divisions should be marked on either edge (*e.g.* eighths, twelfths).

An H *Pencil.*

One or two Set-squares. The edges should not be bevelled, as the set-square has to slide along a straight edge, *e.g.* the edge of another set-square, or the base of the semi-circular protractor.

A Pair of Compasses. It is convenient if the compass will take an ordinary H pencil.

It is doubtful if *dividers* should be used by beginners for any purpose but that of transferring distances; many teachers think that, in measuring lengths or laying off distances, equally good results are obtained by the direct application of the graduated edge (unless expensive dividers provided with screw-adjustment are used).

A Protractor, preferably semi-circular, made of some transparent material; the base not less than 4 inches; graduations running from 0° to 180° each way.

Figures.

In Practical Geometry the final figures should be drawn accurately; but in most cases (*e.g.* drawing to scale, constructions of figures from data, etc.) it is well to begin by drawing a sketch, and marking on it the data. The sketch may be drawn freehand, and the boy should not be allowed to waste time on it; but it should be (1) not too small, (2) reasonably neat; straight lines should look straight, right angles right, and parallel lines parallel. The sketch should be shown on the same paper as the accurate figure.

In Theoretical Geometry an accurate figure will generally be desirable in leading up to a new theorem. But, in writing out known theorems, the pupil should not be expected to draw an accurate figure. What should be demanded is a large figure and a neat figure; greater elaboration is waste of time. The power should be cultivated of drawing neat figures freehand.

Constructions.

A boy should be expected to describe constructions, preferably in his own words; the power of doing this is a valuable part of mathematical training. Examiners should (and probably most of them do) give additional marks for the description.

Examiners should always indicate clearly whether they expect the proof of a construction.

ARITHMETIC AND ALGEBRA.

1. Easy *viva voce* examples should be constantly used in both Arithmetic and Algebra.

2. Great stress should be laid on fundamental principles. It is not desirable, however, to give formal proofs of these principles to young boys, or to expect them to produce proofs in examinations. In arithmetic the fundamental principles should be referred to frequently and should be illustrated by concrete examples rather than by more abstract arguments (*e.g.* $\frac{1}{4}=\frac{3}{12}$. Why do you think that this is true? Because I know that $\frac{1}{4}$ of a shilling and $\frac{3}{12}$ of a shilling each amount to 3*d.*). In algebra there should be the same reiteration of fundamentals, the arguments here being based upon simple instances involving numbers.

3. As far as possible, the rules which a pupil uses should be generalisations from his own experience.

4. Whenever practicable, Geometry should be employed to illustrate Arithmetic and Algebra.

5. Many of the harder rules and heavier types of examples, which examinations alone compel us to retain in a school curriculum, should be postponed.

Long questions in reduction (*e.g.* square miles to square inches) should be avoided. In fact, long and heavy sums generally are of little value; more good is done by setting numerous short sums.

6. Great stress should be laid on the importance of a clear and methodical arrangement of work. From the first, pupils should be taught that "the rough work," as it is usually called, must be done with special care and neatness.

7. Pupils should be encouraged to think out and map out carefully their plan of attack, or their method of reasoning, before starting to work the problem in hand.

ARITHMETIC.

8. In all arithmetic problems pupils should be taught to write down a clear account of their reasoning. A string of figures with no words of explanation should never be accepted as a solution to a problem.

9. All calculations which occur in the solution of a problem should be shown up, and should always be placed as nearly as possible alongside the particular piece of the work to which they refer.

10. The "shop" method of subtraction is recommended.

Example.	927	Say 6 and 1 make 7
	256	5 and 7 make 12
	671	3 and 6 make 9

instead of 6 from 7 leaves 1, etc.

11. In multiplying, pupils should from the very first be taught to begin with the highest digit of the multiplier.

Example.	362
	457
	144800
	18100
	2534
	165434

12. First principles should be carefully taught before vulgar fractions are begun, special stress being laid on factors with index notation, and the use of the signs $+$, $-$, $\times$, $\div$, and of brackets.

13. The two aspects of division (partition and quotition) should be impressed upon the pupil (though it is perhaps better to omit the names); *e.g.* $\frac{12 \text{ pence}}{3}$ asks the question: If 12 pence is divided into 3 parts, how *much* is each part? $\frac{12 \text{ pence}}{3 \text{ pence}}$ asks the question: If 12 pence is divided into parts or shares of 3 pence, how *many* shares?

14. The pupil should not be taught, or allowed, to read $12 \div 3$ as "3 divided *into* 12," a very usual practice which confuses the process with that of "3 divided into 12 parts," a quite different matter. The correct reading of $12 \div 3$ is "12 divided *by* 3."

15. In short division by factors, the reason for the rule for the remainder should be carefully explained, concrete illustration (*e.g.* by counters) being advisable.

16. H.C.F. and L.C.M. should be found by the method of prime factors. The method of finding H.C.F. by alternate division may be postponed, and numbers too large to be dealt with by prime factors should not be used.

The term H.C.F. is preferable to G.C.M., the latter being liable to confusion with L.C.M.

17. Much time should not be spent in teaching complicated fractions.

On the other hand, the pupil should be able to deal with a *complex* fraction, *e.g.*

$$\frac{\frac{3}{5} - \frac{1}{7} \text{ of } 3\frac{1}{2}}{\frac{3}{5} \text{ of } 3\frac{1}{3} - \frac{1}{4}}.$$

18. The pupil should be familiar with the following method of reducing a complex fraction:

To reduce $\frac{4\frac{1}{4}}{3\frac{5}{6}}$. Multiply both numerator and denominator by 12.

$$\frac{4\frac{1}{4}}{3\frac{5}{6}} = \frac{\frac{17}{4}}{\frac{23}{6}} = \frac{51}{46} = 1\frac{5}{46}.$$

19. In order to facilitate the teaching of decimals, a scale divided decimally should be used for the actual measurement of lines; the scale should be subsequently used for the determination of areas and volumes.

In teaching decimals concrete examples should be taken from the metric system.

20. In multiplication of decimals, the noughts should not be omitted. (See example below.)

21. In both multiplication and division of decimals, it is well to insist on a rough approximation being made before the sum is worked. (See examples below.)

22. In multiplication (or division) of decimals, it is recommended that the multiplier (or divisor) be reduced to standard form, *i.e.* to the form in which there is one digit to the left of the point. (See examples below.)

23. In long division, the quotient should be written over the top of the dividend, the point in the quotient being placed over the point in the dividend. The reasoning is identical with that of short division.

Example A.—Multiply 43·2 by 560·8001.

Rough approximation. $40 \times 600 = 24000$.

$$43{\cdot}2 \times 560{\cdot}8001 = 4320 \times 5{\cdot}608001.$$

$$\begin{array}{r} 4320 \\ 5{\cdot}608001 \\ \hline 21600{\cdot} \\ 2592{\cdot}0 \\ 34{\cdot}560 \\ 4320 \\ \hline 24226{\cdot}564320 \end{array}$$

Example B.—Divide ·00019 by ·0346 correct to 3 significant figures.

Rough approximation. $\frac{{\cdot}00018}{{\cdot}03} = \frac{{\cdot}018}{3} = {\cdot}006$.

$$\frac{{\cdot}00019}{{\cdot}0346} = \frac{{\cdot}019}{3{\cdot}46}.$$

$$\begin{array}{r} {\cdot}005491\ldots \\ \hline 3{\cdot}46){\cdot}01900 \\ 1730 \\ \hline 1700 \\ 1384 \\ \hline 3160 \\ 3114 \\ \hline 460 \\ 346 \\ \hline 114 \end{array}$$

$$\frac{{\cdot}00019}{{\cdot}0346} = {\cdot}00549+$$

24. Contracted multiplication and division are not suited for boys of the preparatory school age. Boys should, however, be taught to express results to "so many decimal places," and especially to "so many significant figures."

Recurring decimals, if these are used in the course of a calculation, must be expressed in this curtailed form; no rule for the conversion of recurring decimals into vulgar fractions should be taught, but the converse process can be explained without difficulty.

25. The rule for cube root should be omitted.

26. The following weights and measures are all that need be taught:

BRITISH.

Length	12 inches	= 1 foot
	3 feet	= 1 yard
	22 yards	= 1 chain
	10 chains	= 1 furlong
	8 furlongs = 1760 yards	= 1 mile
Area	12^2 or 144 sq. inches	= 1 sq. ft.
	3^2 or 9 sq. feet	= 1 sq. yard
	22^2 or 484 sq. yards	= 1 sq. chain
	4840 sq. yards	= 1 acre (10 sq. chains)
	640 acres	= 1 sq. mile

Volume	12^3 or 1728 cubic inches	= 1 cub. foot
	3^3 or 27 cubic feet	= 1 cub. yard
Weight	16 ounces	= 1 pound
	14 pounds	= 1 stone
	2 stones	= 1 quarter
	4 quarters	= 1 cwt.
	20 cwts.	= 1 ton
Capacity	2 pints	= 1 quart
	4 quarts	= 1 gallon.

FOREIGN.

Metre, Gram, Litre with the usual prefixes.

Franc and centime. Mark and pfennig. Dollar and cent.

27. In money sums answers should, as a rule, not be required beyond the nearest penny.

The following method of decimalizing money and other quantities is recommended.

To express £3. 14*s.* 7½*d.* as a decimal of £1.

$$\begin{array}{rll} 12\,) & 7{\cdot}5 & \text{pence} \\ 20\,) & 14{\cdot}625 & \text{shillings} \\ & 3{\cdot}73125 & £. \end{array}$$

To express £2·468 in £. *s.* *d.*

$$\begin{array}{rll} £2 \mid & {\cdot}468 & \\ & 20 & \\ \hline 9 \mid & {\cdot}36 & \text{shillings} \\ & 12 & \\ \hline 4 \mid & {\cdot}32 & \text{pence} \end{array}$$

£2. 9*s.* 4*d.*

28. The unitary method should be used, and the rule of three should not be taught.

29. In school entrance examinations the use of algebraic symbols and methods should not be prohibited.

30. Rough checks should be constantly employed in arithmetical work. In many cases, too, a rough estimate might usefully *precede* the detailed work of an arithmetical computation.

In particular, the pupil should not be allowed to regard the mis-placing of the decimal point as a venial error.

31. The Committee consider that the mental training afforded by Arithmetic is largely impaired by the existing tendency of text-books to classify problems and to establish each type as a separate rule.

32. The following subjects should be omitted (some have been mentioned already) or postponed:

Rule of H.C.F. by Alternate Divisions.
G.C.M. and L.C.M. of Fractions.
Very Complicated Fractions.
Conversion of Recurring Decimals into Vulgar Fractions.
Conversion of English Measures into Metric, and *vice versa*.
Cube Root.
Alligation.

Chain Rule.
Bankruptcy.
Compound Interest.
True Discount and Present Worth.
Exchange.
Stocks and Shares.
Equation of Payments.
Contracted methods.
Logarithms.

ALGEBRA.

33. The Committee suggest that beginners should be taught the use of letters to denote numbers, by substituting letters for numbers in suitable illustrations (involving integers) of the fundamental laws underlying Arithmetic and Algebra; they should also be shown how results and processes with which they are familiar in Arithmetic and Geometry can be expressed generally by the use of letters (for example, before beginning Algebra, pupils will have been taught how to solve questions on interest and areas without formulae; from these and similar examples they should be led at an early stage of Algebra to construct formulae for themselves).

34. The fundamental laws of Algebra should be inferred from examples in Arithmetic and Geometry by a rough induction; formal proofs should be postponed.

35. The Committee consider that the idea of a negative quantity should be introduced at the very beginning of Algebra.

36. Instead of the usual difficult substitutions, practice should be given at an early stage in substituting numbers for letters in identities, and in testing the roots of equations, not necessarily of the first degree.

37. The rule for long multiplication and division should be postponed till after problems on x, y equations.

38. The methods used in solving equations should be based on and frequently referred back to first principles (the four axioms); and roots should be tested by substitution.

39. Give constant practice (both oral and written) in solving the simple types:

$$x-4=12\,;\quad 2x=18\,;\quad \frac{x}{2}=6\,;\quad \frac{3x}{2}=5\,;\quad \frac{4}{x}=\frac{1}{2}.$$

In every case insist on an explanation of each step in the solution; also check each solution by substitution, however simple it may be. In written work the final step of the solution should invariably be shown.

e.g. $15x=45.$

Dividing both sides by 15, $x=\frac{45}{15}=\underline{3}.$

Again, $144+x=171.$

Subtract 144 from both sides,

$$x=171-144=\underline{27}.$$

This serves as a useful check to such errors as

$$4x=6\,;\quad \therefore\ x=2.$$

40. The following expressions should never be allowed:

(1) "*Cross multiply.*" Instead use "Multiply both sides of equation by . . . . ."

(2) "*Take over to the other side*" or "*Cancel.*" Instead use "Add (or subtract) . . . . . to (or from) each side."

41. Addition and subtraction of simple algebraic fractions with numerical denominators should be done before solutions of ordinary fractional equations. This should check the tendency to ignore all denominators; a tendency which is greatly fostered by presenting such fractions for the first time in equations.

42. In fractional equations always insist on a written explanation of the step by which the denominators are struck out.

$$e.g.\ \frac{x-2}{5}=\frac{x+1}{7}.$$

Multiply both sides of the equation by the L.C.M. 35.

It is necessary to carefully point out that *both* sides must be multiplied by this number, even when the right hand is only a numeral.

Never allow $\frac{7(x-2)=5(x+1)}{35}$.

43. Young boys should not be encouraged to clear of fractions and ***also*** remove all brackets in one step; for in the majority of cases this results in mistakes in sign and mistakes in multiplication.

44. In testing roots, the pupil commonly adopts the following illogical arrangement, which should be prohibited.

Example. $\frac{x-1}{3}+4=\frac{3x-2}{5}+3.$ .....(1)

Solution. $x=4.$ .....(2)

Proof. $\frac{4-1}{3}+4=\frac{12-2}{5}+3;$ .....(3)

$\therefore\ 1+4=2+3;$

$\therefore\ 5=5.$

The objection to the above is that the statement of line (3) is untrue if the solution (2) is incorrect.

Each side should be taken separately, and reduced to its simplest terms. Thus:

Proof. If $x=4$,

$$\text{L.H.S.}=\frac{4-1}{3}+4$$

$$=1+4$$

$$=5,$$

$$\text{R.H.S.}=\frac{12-2}{5}+3$$

$$=2+3$$

$$=5;$$

$\therefore\ x=4$ is the correct solution.

45. In solving simultaneous equations, the equations should be numbered and details of the solutions written out at each step, *e.g.* in solving

$5x-4y=6,$ .....(1)

$8x+3y=7,$ .....(2)

multiply equation (1) by 8 and equation (2) by 5, subtract and so on.

$$\left.\begin{aligned}5x-4y&=6\times 8\\8x+3y&=7\times 5\end{aligned}\right\}$$ should never be allowed.

Practice in finding the remaining unknown, given one of the two, is often a useful introduction to simultaneous equations.

46. H.C.F. and L.C.M. of monomials should be practised at the same time as H.C.F. and L.C.M. of numerals (in their prime factors).

H.C.F. should always be found by factors, harder cases not being suited to the preparatory school stage.

47. *Involution and Evolution.* Pupils should be given a clear idea as to the reasons for the index rules of multiplication and division; reference should continually be made to first principles, *e.g.*:

$$(a^3b)^2=a^3b\times a^3b=a\times a\times a\times b\times a\times a\times a\times b=a^6\times b^2.$$

48. Factors should be tested sometimes by multiplication, sometimes by substitution of some numerical value.

Factors of the type $3x^2-9x$ should not be neglected.

49. Multiplication and division of fractions involving binomial expressions should be introduced by repeated revision of similar types in monomials and arithmetic. Instead of "cancelling," divide numerator and denominator by the same factor.

50. Problems should be introduced as soon as the simplest equation can be solved. Details should be written, showing how the equations have been built up; the equations alone are not enough.

51. Stress should be laid on the application of algebraical symbols to simple problems, *e.g.* how long do I take to cycle x miles at y miles per hour?

52. Algebraic work should be checked by the consideration of simple special cases and the substitution of numbers.

53. *With regard to graphs*, the Committee find that there is much uncertainty as to the exact meaning in which the term is used. They suggest that, in an ideal course of teaching, the solution of simple simultaneous equations should be *illustrated* by the drawing of graphs. For this purpose, the first type of graph to be introduced should be of the form

$y=ax+b$, in which a and b are numerical coefficients (e.g. $y=3x-2$).

The main point to make clear is the connection between the solution of two simultaneous equations and the coordinates of the intersection of the corresponding graphs. The Committee believe that if this idea is grasped, the chief object of the teaching at this stage has been attained; and that it is undesirable, at this stage, to lay stress on the *practical employment* of graphs in solving equations or other problems.

It may be found that the graphing of a few sets of statistics is useful in leading up to the idea of graphing the pairs of numbers which satisfy such an equation as $y=3x-2$.

In practice it is by no means easy to bring home to young boys the idea referred to in the preceding paragraphs. The Committee consider it very desirable that the teaching of this subject should be in the hands of a teacher who is well acquainted with the further developments of mathematics.

54. The following subjects should be omitted or postponed:

H.C.F. by alternate division.
Complicated fractions.
Fractions with decimal coefficients in the denominator.
Literal equations.
Cube root of polynomials.

LETTER TO EXAMINING BODIES FROM THE MATHEMATICAL ASSOCIATION.

GENTLEMEN,—I am instructed to lay before you a copy of the following resolution passed at a meeting of the Mathematical Association Committee on September 28th :

"That, in view of the doubt felt by many teachers as to the practical use and educational value of the ordinary rules for contracted multiplication and division, examining bodies be requested not to insist on the use of these methods ; and that such bodies as agree be requested to publish their decision."

My committee will be glad to hear your decision.

I enclose a few copies of this circular, hoping that you will lay the above resolution before your examiners.—I am, yours faithfully,

A. W. SIDDONS,
Hon. Sec., Mathematical Association Committee.

HARROW-ON-THE-HILL,
October 21st, 1907.

THE INTRODUCTION OF THE MATHEMATICAL IDEA OF INFINITY

BY W. H. YOUNG, M.A., Sc.D., F.R.S.

(*Address delivered before the British Association, at Leicester*, 1907.)

I.

IT is a commonplace that ideas which seem abstruse or even incomprehensible to the learned of one generation are easy and familiar to the children of the next. This is, and must be so, with the mathematical idea of infinity. While there are still scientific people who regard the attempt to conceive anything beyond a finite number of things as visionary, I know of at least one child of 10 years old who dallies lovingly with the idea of infinity and the symbol ∞. To him, as to me, the idea of the number of reflexions of two looking-glasses, placed facing one another, is as definite and as unavoidable as that of the number of ants in an ant-hill.

The concept of a number is distinct from the manipulation or arithmetic of numbers. The idea of a number involves that of one number being comparable with other numbers, that is to say, it involves, more or less directly, the ideas of greater, equal and less, but not of the processes by which we answer the question—Is this number greater than that? Thus to recognise that there is a certain definite number—the number of ants in an ant-hill, and another definite number—the number of reflexions of two looking-glasses, placed facing one another, although neither of these numbers can be determined by counting, is one thing, but it is a step farther to realise that the second of these numbers cannot be reached or surpassed, by addition

and multiplication, while the number of ants in an ant-hill may be calculated more or less accurately by observation and arithmetic. It is indeed clear that the size of an ant is more than $\frac{1}{10,000}$ cubic inch, and that of an ant-hill less than 100,000 cubic inches, so that the number in question is less than $100,000 \div \frac{1}{10,000}$, that is, less than a thousand million.* But, barring the imperfections of the mirrors and of our eyesight, there are a thousand million reflexions and more. Every integer is too small for the number of reflexions, and so we say that *integers are finite numbers*, while the number of reflexions is *an infinite number*.

Each reflexion corresponds to a definite integer; there is the first, the second, the thousand-millionth, and so on, and so we call the number of reflexions *the countably infinite*, to distinguish it from other infinite numbers, of which more anon.

These infinite numbers, and particularly the countably infinite, played an unrecognised part in the mental life of thousands before Georg Cantor put them and their properties lucidly and systematically before the world. The mathematical idea involved is clear and precise, and my present object is to give some indication how it may gradually be elucidated in the ordinary course of a modern education.

II.

There are three methods of dealing with mathematics:

1. The Logical Method,
2. The Formal Method,
3. The Practical Method.

The second of these has held the field for a long time in English mathematical training, and to it is due the undoubted facility many of our schoolboys and undergraduates possess in the manipulation of numbers and symbols. But it has absorbed an undue amount of attention. I myself stand for both the logical and the practical methods.

The practical view of mathematics and mathematical training is one that is now rapidly gaining ground in this country, so that it is the less necessary to plead for it here. For this reason it is with the first point of view, the logical, that I propose chiefly to deal in the present remarks.

I remember a story of J. J. Thomson's of a pupil of a practical turn of mind who took no interest in his mathematical work until Thomson took two marbles and made them collide. By that simple trick he won the interest of his pupil, who would otherwise have neglected his mathematics, and, Thomson thinks,

* Actually very much less, I believe.

probably have failed in the Tripos; as it was, he ultimately came out among the Senior Optimes.

So I believe some of the best minds have been lost to English mathematics by neglect of the logical method of treatment. I can parallel J. J. Thomson's story by one of my own of a man who told me he had given up mathematics after a year at Cambridge because of $\sqrt{-1}$. He said its use involved a contradiction of all previous axioms.

You must follow the *spirit* of J. J. Thomson's lead if you mean to catch the logically minded boy for mathematics. You must not wear him out solving quadratic equations, or working out by multiplication the number of bricks in a brick wall, or determining the cubic contents of a box by the method of weighing. His mind craves for more than these things, reasons, ideas; his imagination hungers for the unknown.

The moral effect of the logical method must not be overlooked. The second and third methods are alike in this—that they avoid difficulties; the first method alone looks facts in the face.

III.

The different methods are well exemplified in Geometry, and this will bring us to the subject of my paper.

The practical method involves drawings and diagrams, perspective, actual construction of solid models and of sections of solids, processes much neglected in England, but now coming in.

The formal method is the one still to some extent in possession of the field. In it results are regarded as interesting solely for their own sake, and theories only so far as they are productive of new results.

The logical method is that in vogue in Italy and, to a greater or less extent, all over the continent of Europe. On the one hand it concerns itself with the framing of a consistent code of axioms, and on the other it classifies the results obtained on a methodical and intelligent system.

Euclid, though of great interest to the antiquarian, represents a point of view unsatisfactory to all three schools of thought: not enough variety of results for the formalist, not enough logic for the modern mathematician, while the upholders of the third method complain of the thoroughly unpractical point of view exemplified by the proof that a straight line cannot meet a circle in more than two points, without making use of obvious considerations (of symmetry, for instance) that occur to the practical mind.

Infinity is never mentioned in Euclid. The idea is indeed of much later date. It was, I believe, in France that the word *infinity* intruded itself into geometry; Desargues, Descartes and Pascal seem to be responsible for it. It arose, not from

philosophical speculations as to the ultimate, but as a help to grasp the anomaly presented by parallel lines in the theory of straight lines and their intersections.

The way I like to put it is this: The introduction of the terms "point at infinity" and "line at infinity" is merely a convenient symbolism. When we say, "These two straight lines have in common a point at infinity" we use the expression merely as standing for "These two straight lines are parallel."

It is easy to explain, even to a child, what we gain by this alternative mode of expression. That the whole subject of projective geometry, with its carefully selected code of axioms, takes its rise, in a certain sense, from this convention, may perhaps appeal to few who have been educated on this side of the Channel; but the fact that it is possible to replace half a dozen enunciations and as many different proofs by a single theorem* cannot fail to startle the student and to engage his interest. In particular, the engineer will soon perceive that he is enabled to remember a number of facts of practical value to him, which, when presented in an unsystematic manner, were too multitudinous for him to manipulate.

There is no hypothecation here of the *existence* of points, or lines, at infinity. There is, in fact, no new idea introduced at this stage at all. In the language of the theory of sets of points, our actual conception of the geometrical universe is that of an entirely open region without a boundary, or what is sometimes called *a domain*. Our senses limit the extent to which we can probe that domain, but our thought wanders unbounded throughout its depths. The imagination may, perhaps, like to carry over the symbolic language introduced for convenience, and to picture to itself the universe as a closed set of points with a plane at infinity. If, however, we have grasped the fact that the language is merely a conventional one, the student, when he comes to Theory of Functions of a

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*Instead of saying:

"Two straight lines in a plane have a common point, unless they are parallel" } we say { "Two straight lines in a plane always have a single common point."

Or instead of the following:

"Three planes have a single common point, unless they have none at all, or a straight line common. If they have no common point, there are three cases: (1) they are all parallel; (2) two are parallel; (3) no two are parallel, but the three straight lines of intersection of the planes in pairs are parallel." } we say { "Three planes always have one common point, and, if they have two, they have in common the whole straight line joining those two points."

Examples might be multiplied, but it is not only that we have here a more handy form of statement, we are enabled to discuss simultaneously a number of special cases, and so to avoid the tedious repetition of arguments in which no new idea is introduced, and which therefore render the subject unnecessarily tedious. A good example of this is the celebrated theorem that, if the corresponding sides of two triangles ABC and $A'B'C'$ meet on a straight line, then the rays joining corresponding points, AA', BB', CC', meet at a point, and *vice versa*.

Complex Variable, will have no difficulty in adopting a different symbolic language, and, if he gives play to his imagination, may prefer to think of his unlimited domain as a three-dimensional sphere, closed by a single point at infinity.

IV.

I have placed geometry first, because of the vivid appeal which it makes both to the practical mind and to the imagination for the introduction of the infinite. But it is not through Geometry that the mathematical idea of the infinite is, in the natural course of a modern education, unavoidably brought before the mind even of the child.

A circulating decimal, with all the difficulties involved in conceiving an infinite number of *processes*, is forced upon the pupil from the formal or the practical standpoint, when he attempts, as he is bound to do, whatever the system * under which he is brought up, to express $\frac{1}{3}$ as a decimal fraction. Later on the theory and practice of logarithms brings the same logical difficulty.

It is desirable that the boy, or girl, should be given a rigid proof of, for example, the theorem that there is no number, integral or fractional, whose square is 2. The old Greek proof will serve.

If there were such a number, we could denote it by $\frac{m}{n}$, where m and n are integers without a common factor. Then m is either odd or even. But the square of an odd number is odd, while $m^2 = 2n^2$,

and is therefore even; thus m must be even.

Put then $m = 2p$,

where p is an integer. Then $4p^2 = 2n^2$,

that is, $2p^2 = n^2$,

so that n^2 and therefore n is also even, contrary to the fact that m and n, having no common factor, are certainly not both divisible by 2.

Therefore there is no such number $\frac{m}{n}$. Q.E.D.

In the same way a proof should be given that there is no number, integral or fractional, which is the logarithm of 2 to the base 10. For, if $\frac{m}{n}$ were such a number, it would mean that

$$2 = 10^{\frac{m}{n}},$$

so that $2^n = 10^m$.

* I am glad to say that circulating decimals have no longer the prominence in the ordinary curriculum that they enjoyed when I was a boy. We must not, however, entirely banish them, or fail to assign to them their proper place, when devising our logical scheme of education.

But 2^n ends in either 2, 4, 8 or 6, and cannot therefore end in a 0. Thus this is impossible, and there is no such number $\frac{m}{n}$. Q.E.D.

At this stage I believe that the right point of view is not the existence, but the non-existence of the irrational * number, and that the use of the term is, as in Geometry, merely a convenient convention. When I say, "There is no number, integral or fractional, whose square is 2," and when I say, "There is a number, an irrational number, whose square is 2," I mean the same thing. If the latter statement is more suggestive, it is because, by our convention, it is equivalent to the positive assertion that, *to any required degree of accuracy, any rational number within a certain interval will serve our purpose.*

The importance of this convention in relation to the concept of length should be clearly brought into view. It does not follow that the lengths of two objects, taken at random, are to one another as two numbers, unless and until this convention has been made. It is invariably true, of course, that two lengths, as measured, however accurately, are in the ratio of two integers, but these integers will in general vary according to the delicacy with which the measurement was made. The quantitative concept of length indeed itself involves the idea of an infinite number of processes; still more is this true of the ideas of area, volume and density.

V.

The familiar measuring rod or rule connects with points on the straight line the idea of definite numbers, their distances from the zero point of the rod. Certain of these numbers are indicated on the straight edge by marks; the integral points 1, 2, 3, ... most certainly, while probably the rod is subdivided into tenths, perhaps into hundredths, and a further subdivision is usually effected by the eye. The points so identified by continued subdivision will be the end-points of intervals; any other point must be thought of as the limit of an interval, which is at first a marked interval and gets smaller and smaller till it shrinks up to a point.

Thus, for instance, the point $\frac{1}{3}$ is never obtained by the decimal subdivision. It lies first between the points ·3 and ·4, that is in the interval (·3, ·4). The next subdivision may or may not be feasible, but we only have to imagine that we look through a magnifying glass which enlarges 10 times, and it will be clear to us in what interval our point will lie. For

* A rational number is the general term for any integer or fraction. An irrational number is then, in the first instance, the negation of the rational, and is subsequently used for any of the new numbers conventionally introduced.

that which we must add to ·3 to get $\frac{1}{3}$ is $\frac{1}{3} \cdot \frac{1}{10}$, that is $\frac{1}{3}$ of the smaller interval (·3, ·4), so that when we look through our microscope the point $\frac{1}{3}$ will seem to have the same position in the interval (·3, ·4) as it originally had in the interval (0, 1) as seen by the naked eye. Thus it will lie in the third interval when the small interval (·3, ·4) is subdivided into hundredths, that is it will lie between ·33 and ·34, or in the interval (·33, ·34).

This is the same at every subsequent stage, so that the next interval in which $\frac{1}{3}$ lies will always be the third subdivision of the preceding interval; in this interval, if we have a sufficiently strong magnifier, our point will always seem to lie.

The intervals we get then, which are at first marked intervals and afterwards only imagined, are, (·3, ·4), (·33, ·34), (·333, ·334), (·3333, ·3334), ... and so on *ad infinitum*; these intervals shrink up to the point $\frac{1}{3}$. Having once made this clear to our minds, the idea that intervals lying inside one another, and getting indefinitely small, do actually shrink up to a point *wherever the intervals may be located on the straight line* is instinctive: the uniformity of the unmarked straight line demands this.

The point to which the intervals shrink up is called *the limiting point* of the intervals, or of the end-points of the intervals.

The distance of a point P from the zero point O may, as in the case of the point $\frac{1}{3}$, be a number, integral or fractional—a rational number—but this need not be the case. Suppose OP is the long side of a right-angled triangle, whose other sides are each of length unity. Here, in truth,

$$OP^2 = 2,$$

so that, as we saw, the length of OP is not a rational number, but, to any required degree of accuracy any rational number within a certain interval will serve our purpose. We find ourselves constrained to say that the distance OP is *an irrational number*, and to call P *an irrational point*. At this stage we begin to regard irrational numbers as on a par with rational numbers, and to conceive the straight line as made up of rational and irrational points.*

VI.

What are the odds that, if we take a set of intervals located at random and shrinking up to a point, that that point will

*The irrational point P, corresponding to the irrational number $\sqrt{2}$, is constructed without the use of intervals as the intersection of two straight lines. It cannot be proved that this is the case with *any* irrational number, and it is known that in the case of some irrational numbers, *e.g.* the length of the circumference of a circle of diameter unity, the construction (so called "squaring of the circle") cannot be carried out by means of ruler and compasses. It is assumed as an axiom in the theory of sets of points that every irrational number, by whatever mathematical process it is obtained or defined, *determines* a point on the straight line.

be a rational point? Here you have a pretty problem in infinite numbers. There is an infinite number of rational points, but there is also an infinite number of irrational points; yet the question has a definite answer, just as it would if the numbers in question were finite.

The answer is that you are tolerably certain to hit upon an irrational number; for there are, to put it in schoolboy language, heaps more irrational points than rational ones. I mention this to shew how one is led on from one infinity to another, and to illustrate what I said at the beginning about other infinities beside the countably infinite.

The rational points are countably infinite, as you easily see, for you can arrange them in the following way:

$$\tfrac{1}{2}, \quad \tfrac{1}{3}, \tfrac{2}{3}, \quad \tfrac{1}{4}, \tfrac{3}{4}, \quad \tfrac{1}{5}, \tfrac{2}{5}, \tfrac{3}{5}, \ldots$$

so as to get them all, and each in its definite place, so that you can say which is the first, or the twentieth or the thousand-millionth, just as we could in the case of the two mirrors.

Now, it is not hard to shew, by the use of decimal fractions, that the irrational points are *more than countably infinite*; there are, of course, countably infinite sets of them, but no countably infinite arrangement is possible that takes in all.

The two infinite numbers we have here learnt to know are commonly called a and c, a being less than c. There is nothing indefinite or hazy about these numbers; they are recognisable, and they constantly crop up; c is, by the way, startling as it may seem, also the number of points on a straight line, or in a plane, or in space.

VII.

The concept, above referred to, of a point P being *a limiting point of a set of points*, is a very valuable one. The sole and only criterion is that every interval, however small, containing P, encloses a point of the set, distinct from P itself.* When, and only when, this is the case, P is said to be a limiting point of the set.

This does not seem to be a complicated notion, and it is simple enough to detect whether, or no, any given point P is, or is not, a limiting point of a given set. On the other hand, given only the set, the identification of its limiting point, or one of its limiting points, involves an infinite number of processes.

It is only by degrees that the idea of an infinite number of processes can be rendered familiar. If this has been done, the notion of *a limit* will no longer present serious difficulties. A limit is merely the equivalent in arithmetic of that which

* P may, or may not, belong to the set, according as the set is defined. Thus any point is a limiting point of the rational points, but it only belongs to the set if it is itself a rational point

we called *a limiting point* in geometry, the identification of numbers with points effects this. Whatever numbers we have to deal with, for instance the values of any number of areas, we can plot down on a straight line, using a suitable system of units;* if there are only a finite number of points, there will be no limiting point; but, if the points to be plotted are infinite in number (in which case, they cannot, usually, be all actually plotted down) they will certainly have at least one limiting point, which may, or may not, be capable of being plotted down, but can certainly be determined within the limits of vision, which is only another way of making the old statement about the limits of accuracy.

Too often, at present, the pupil comes openly face to face with the idea of infinity first in connection with infinite series, which are defined by means of limits,† *e.g.*

$$\underset{n=\infty}{Lt}\left[\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2^n}\right]=1,$$

or

$$\frac{3}{10}+\frac{3}{100}+\frac{3}{1000}+\ldots=\frac{1}{3}.$$

Generally the sum to infinity of geometric series, such as the above, is sprung upon him, after learning the rule for summing a finite number of terms. In other words, he is supposed to have intuitively a notion of what is meant by a "limit," and, in this way, to be able to explain to himself the idea of a infinite number of terms, whereas almost as soon as he is capable of abstracting the idea of number at all, the concept of infinity should begin to play its appropriate part in his mental drama of the universe.

VIII.

The number of familiar notions which are absolutely dependent on the idea of a simple limit, or limiting point, is large. It is no exaggeration to say that few people have more than a vague idea what they mean by length, area, volume, mass, moment of inertia, current.

To take the notion of area as typical. It is easy to define the area of a triangle, and to prove that it is "half the base × altitude." We naturally say that the area of any number of non-overlapping

* For instance the area 2 sq. metres may be plotted down as the point whose distance from the zero point is 2 millimetres, and so on.

† Plotting down the points $\frac{1}{2}$, $(\frac{1}{2}+\frac{1}{4})$ or $\frac{3}{4}$, $(\frac{1}{2}+\frac{1}{4}+\frac{1}{8})$ or $\frac{7}{8}$, and so on, taking each time one more term into the sum, we get on the straight line points, which soon, within the limits of vision, are indistinguishable from the point 1; this point 1 is in fact the limiting point of the set, and the number 1 is the limit of the set of numbers. The successive sums of the second series are plotted down as the left-hand end-points of the intervals of § VI., which, as we saw, shrank up to the point $\frac{1}{3}$, and the whole infinite series is only another way of writing the circulating decimal $\dot{3}$.

triangles is the sum of their areas, and so get an expression for the area of any polygonal figure. But what do we mean by the area of a circle, or of a figure of more irregular form?

It is proved in elementary books that, if we inscribe a polygon in a circle, and circumscribe one about the circle, the difference between the areas of the two polygons may be made as small as we please, and depends only on the smallness of the sides of the polygons. In other words, the areas of the two polygons will agree to more and more decimal places, the shorter the sides are constructed, in which case they become, of course, more and more numerous. We thus get a definite limit defined, which is not itself the area of any one of these polygons, and depends entirely on their being inscribed in and circumscribed about this particular circle. This limit is what we call the area of the circle.

This is a notion precisely of the same kind as that of length, triangles and polygonal regions taking the place of the rational lengths. If we plot down these areas on a straight line, we get, as before, a series of intervals—between the areas of the inscribed and circumscribed polygons—intervals which get smaller and smaller and finally shrink up to a point.

This definition of the area of a circle suggests that the same kind of definition would do for more irregular figures. But are we sure that the difference between the areas of the inscribed and circumscribed polygons would always diminish indefinitely?

Usually this is so, but it is stimulating to realise that it is possible to find a figure with a crinkly boundary where this is not the case. Clifford is certainly not the only Englishman who has revelled in the possibility of the incredible. Incredible as it sounds that a curve, however crinkly, should take up so much room that the areas of the inscribed and circumscribed polygons never agree beyond a certain point, it has been rigidly proved to be true. When, and only when, the boundary of the region considered is not of this nature, we do get, as in the base of the circle, a definite limit, which is then called the area of the region.

The definition of volume is of precisely the same nature as that of area; that of mass is somewhat more complicated; the other familiar notions I mentioned, as well as that of velocity, involve the idea of a simple limit. It is the same thing to say that all these notions involve the conception of a limiting point on a straight line.

Just in the same way as we may describe a portion of a plane cut out by a curve, so as to have no area, so we can isolate in our minds a portion of space which has no volume, and imagine a suitable surface drawn in a body so that the portion inside has no mass. We may even conceive of the

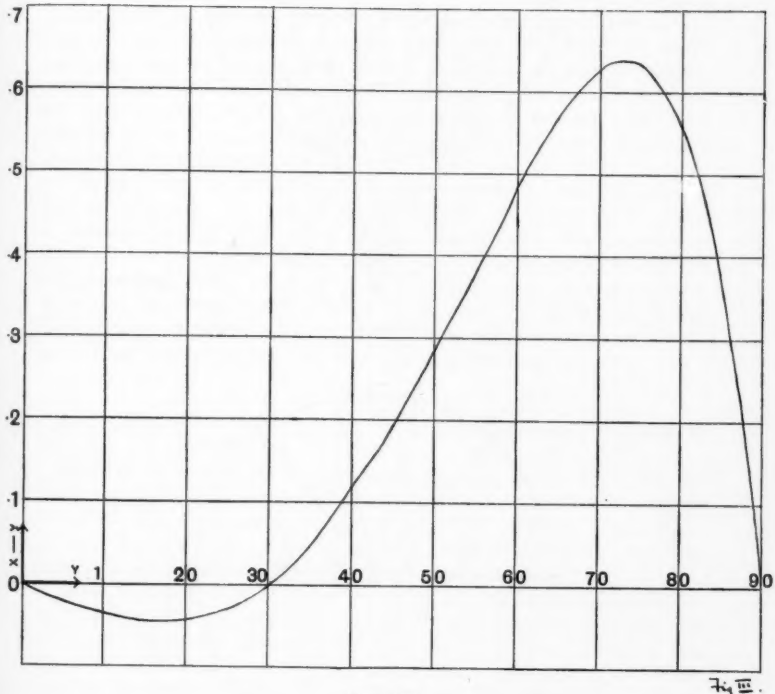


FIG. III.

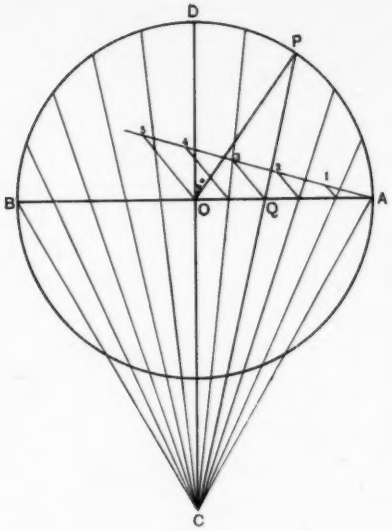


FIG. I.

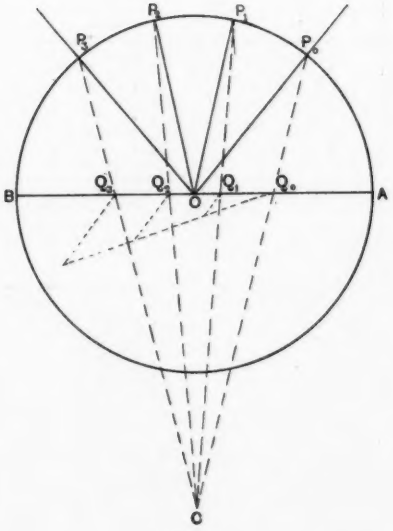


FIG. II.

To be bound up opposite page 98.]

possibility of a body moving continuously in such a way that no point of it has any velocity.

IX.

From limiting points on the straight line, it is a natural step to limiting points in the plane and in space. These are still more interesting because they introduce new ideas and new processes of thought, but they are as easy of conception as limiting points on the straight line. We have, instead of intervals shrinking up to a point, small plane domains, or small solid domains in space, lying inside one another and shrinking indefinitely in size. We already had an example of this, in the case of our two mirrors; on the face of one of them the reflexions are plane domains, lying inside one another, and shrinking up to a point.

The practical importance of the ideas so easily obtained cannot be exaggerated; for the idea of a limiting point in a plane is concomitant to that of a double limit,* and the conception of a double limit is a real difficulty in analysis and in applied mathematics. Still more is this the case with a triple limit, which is however concomitant to a limiting point in space. The proper comprehension of these ideas enables us to be lucid as well as precise in dealing with these important subjects.

X.

Let us now utilise the idea of a limiting point in space to clear up our notions about *the centre of gravity.*

We all use the term centre of gravity, but what do we mean by it? The centre of gravity, you tell me, is a definite point in a body through which the weight always acts. How do we know that there is such a point? Is there, in fact, always such a point? Anyhow, how do we find it?

Now I do not mean to enter here into the difficulties introduced by the fact that gravity does not really act in parallel lines, and so forth. Assume that gravity does act in parallel lines, if you like, or, as I prefer to do, let us use the term *centre of mass*, instead of centre of gravity, the questions just asked still present real difficulties.

It is easy to see by symmetry that a cube or a sphere of uniform matter has its centre of mass at its centre of symmetry. It is no hard practical problem to find the centre of gravity of a uniform rod, or a billiard cue. We have an easy rule for finding graphically or numerically the centre of mass of any two bodies together, provided we know their masses, and their centres of mass separately. But how are we even to define the centre of mass of a body of irregular form and variable specific gravity?

* This is quite distinct from two simple limits in succession.

If anyone says "we define it by triple integration," * it is clear that this is not a definition likely to appeal to the schoolboy, or the man in the street. But the idea involved is not really occult, it is merely that of a limiting point in space.

Imagine space divided up by parallel planes into small equal cubes. Certain of these will be entirely filled by the body under consideration, certain others will be partially filled, and the rest will contain no point of the body. The cubes containing points of the body will, however, not necessarily be uniform, so that we do not know their centres of mass. We imagine however the mass inside each to be distributed uniformly, so that the centre of mass of each is its centre of symmetry, and we find the centre of mass G_1 of all the cubes which are completely filled with the matter of the body under consideration. We also find the centre of mass G_2 of all the cubes which contain any points whatever of the body.

We now subdivide space into cubes of $\frac{1}{8}$ of the size, by interpolating planes between each pair of dividing planes, and repeating the process we get two new points G_3 and G_4.

Proceeding thus *ad infinitum* we obtain a countably infinite set of such centres of mass $G_1, G_2, G_3, G_4, G_5, G_6, \ldots$

Now it is one of the first things we have learnt about limiting points that every infinite set of points has at least one limiting point. Thus these centres of mass have at least one limiting point, and Nature provides that, just as we do not meet casually with bodies which have no area or volume, so these centres of mass are found habitually to have only one limiting point; moreover, this limiting point is found to be the same however the dividing planes have been located. This limiting point G is called the centre of gravity, or more correctly the centre of mass, of the body.

You see at once that you have here the same hypothesis as to the existence of the centre of mass as in the case of area, volume, or any idea involving limits, but the concept is hardly more complicated than any of the others already touched upon.

XI.

Before closing, there is one familiar idea which I must recall to your minds. It is the concept of time.

There is no idea which is more familiar or more unavoidable in modern life. For ordinary purposes a Waterbury watch or a Dutch clock will perhaps suffice to give us a vivid idea of the nature and use of time. But any really fine observational work brings us face to face with the actual problem, "How

* It may be added that such a definition shirks the real difficulty. The triple integrals may indeed not exist.

are we to make our time-keepers accurate?" and with the logical question, "What is time, and how is it measured?"

That the first question depends on the second is evident. That the answer to the second question involves in a number of ways the idea of infinity is the reason why it appears so inscrutable to most ordinary people who have once tried to understand it. The measurement of time involves the mathematical ideas of distance and velocity (or angle and angular velocity), approximation, disturbance, curve, and a number of other notions summed up in the term limit; notions which will not seem hard or abstruse to people who have never avoided, but courted, the mathematical idea of infinity.

I have done. Allow me to sum up in one sentence the gist of what I have said. The mathematical idea of infinity and of an infinite number of processes is unavoidable, nor is it desirable that it should be avoided. It has been rendered precise, and its contemplation tends to clearness of thought. Its introduction from the earliest stages as a recognised object of thought is in every way to be encouraged.

PERSPECTIVE THROUGH THE STEREOSCOPE.

(*Concluded from* p. 122.)

Slide IX. (fig. 7). Here the Ground Plane is rotated about the Intersection carrying P with it along a circular arc PP' whose centre is M. At the same time E and J are rotated in vertical circles whose centres are I and O through angles EIE', JOJ' each equal to PMP'. Since $E'I$ is parallel to MP', they must lie in a plane cutting the Picture Plane in IM. Also since $E'P'$ is in the plane containing $E'I$ and MP', the point in which $E'P'$ cuts the P. Pl. lies in IM and divides it in the ratio of $E'I : MP'$, *i.e.*, of $EI : MP$.

Hence $E'P'$ cuts the P. Pl. in p. Or thus: Since EIE', PMP' are similar figures with corresponding sides parallel, therefore EP, IM, $E'P'$ meet in a point.

The same applies to any other point in the G. Pl. Hence

Theorem (5) *If the Ground Plane and the eye be both rotated in the same sense through equal angles, the first about the Intersection and the Second about the I-line, the picture remains unaltered.*

Hence E may be rotated into the Picture Plane itself into a position above or below the I-line at a distance equal to the Distance of the Picture EI, as we have previously shewn.

From the previous investigation it follows that the I-line is still the Vanishing Line of the inclined plane in the position shewn and *with the eye at E'*.

Hence the Vanishing Line for the same inclined plane with

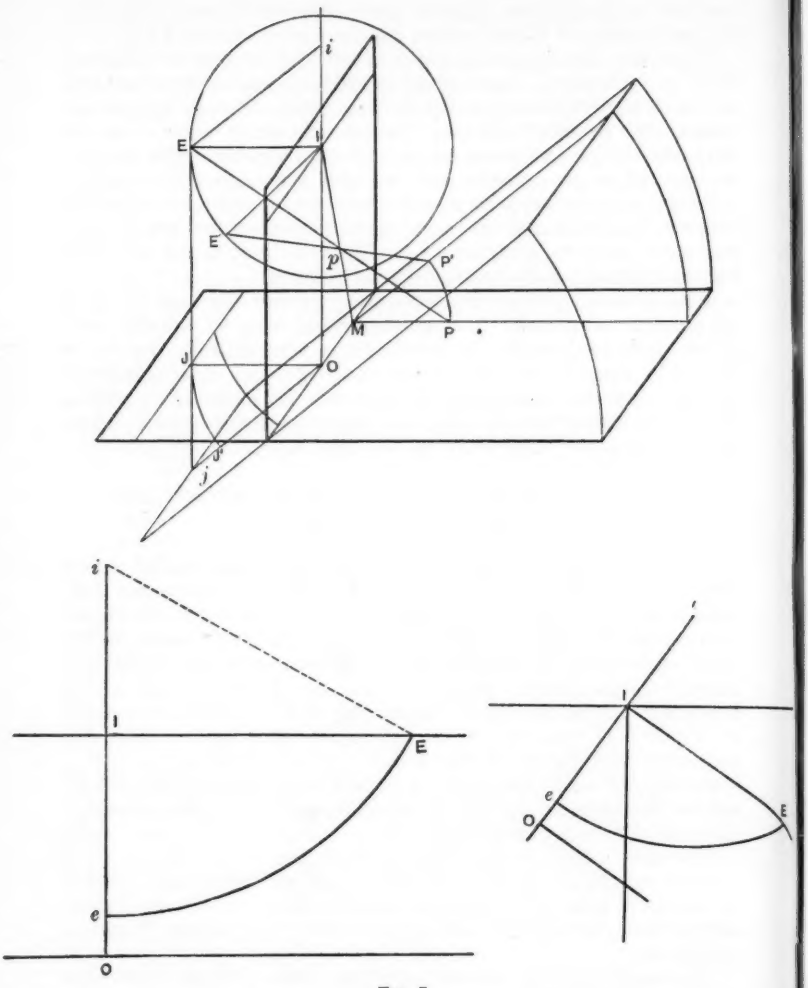


FIG. 7.

the eye at E is a line parallel to the Intersection through a point i in the P. Pl. such that Ei is parallel to $E'I$.

Hence to find i take E on the I-line equal to the Distance of the Picture and make the angle IEi equal to the inclination of the plane in question to the horizon (so that IiE is the

inclination of the plane to P. Pl.). If Ie be taken along iI equal to iE, e will have the same property as the rotated E for the previous constructions, *i.e.*, e will be the centre of plane perspective of the original figure rotated into the P. Pl. and its *image* in the P. Pl.

There will be a corresponding j, such that Oj is equal to, and in the same sense as, ie.

Slide X. (fig. 8). Here P is a point in space, Q, its projection on the G. Pl., KL, MN the projections of EJ, PQ on a Third Plane

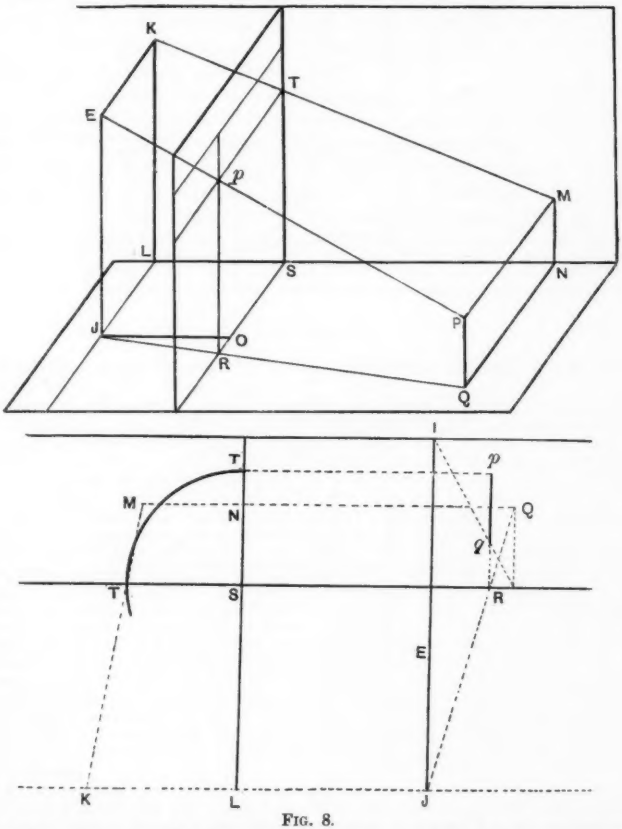


FIG. 8.

at right angles to the Intersection. Since EK, MP are parallel, they lie in a plane cutting the Picture Plane in a line Tp parallel to the I-line, that is, the line corresponding to MP is Tp.

Again, since EJ, PQ are parallel and in planes parallel to the Picture Plane, they lie in a plane cutting the Picture Plane in a line Rp perpendicular to the Intersection through R where it is cut by JQ. Hence p is the point at which Tp crosses Rp.

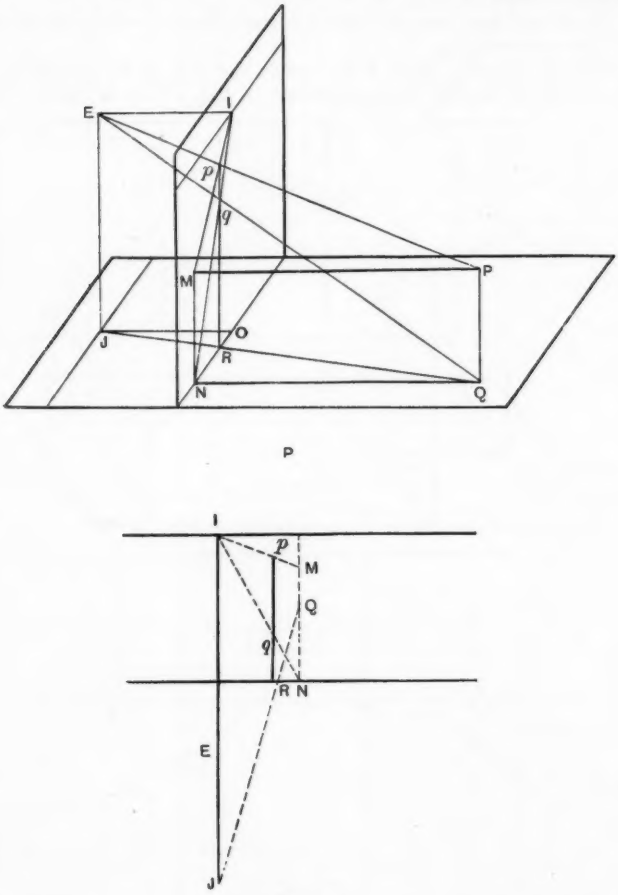


FIG. 9.

Hence the following construction *in plano*. Suppose the Third Plane folded about LN into the G. Pl. and then both together to be folded about the Intersection into the P. Pl., and draw K, M, N, Q in the positions they would then have from

the known magnitude and position of PQ. Note that the distance OS is arbitrary; that (Q, M) are the 'plan and elevation' of P with reference to the G. Pl. and the Third Plane, LS being the ordinary 'xy line'; and that similarly (J, K) are the 'plan and elevation' of E in its original position. To find T join KM cutting OS in one of its positions and inflect the distance ST along SN. Join JQ cutting the Intersection in R and complete the rectangle $TSRp$.

Slide XI. (fig. 9). Here again P is a point in space, Q, its projection on the Ground Plane. The corresponding points are determined by finding the points p, q, at which the lines joining I with their projections (as in I) cut the line Rp on the Picture Plane corresponding to PQ.

In Slide XII. (not reproduced here) the plane of the card is that of the Picture, and two vanishing scales are drawn on it, one (XOY) for the Ground Plane, another (XOZ) for some plane perpendicular both to the Ground and Picture Planes. The position of a point being given by its distance (2, 3, 4) from these planes, in other words by its co-ordinates, the diagram indicates how the point corresponding to it in the Picture can be at once determined.

E. M. LANGLEY.

REVIEWS.

Cartesian Plane Geometry. Part I. **Analytical Conics.** By CHARLOTTE ANGAS SCOTT, D.Sc. (Dent & Co.)

This introduction to Analytical Geometry appears at a most opportune moment, when there is a general desire to see the dead bones of Mathematics come to life again in our schools and higher places of education. It differs as much from the ordinary elementary text-book as light from darkness; and we sincerely hope that it will become generally adopted. For the success of such a book there are two requisites: first, that it should become generally known to teachers, and second, that examining boards should make it a rule to accept only questions on what is important, and reject questions on what is unimportant.

For years we have suffered from the evils of over-elaboration, although we now recognize that it is an evil, and are making a determined and successful effort to introduce a better system. The reader of this book will not waste his time over details and unimportant things, but will find every page, with the exception of the last chapter, a direct help to future progress. Its distinctive feature is the systematic use of line (tangential) co-ordinates concurrently with point co-ordinates. This, in our opinion, simply presents the subject in its proper light, and gives it completeness; it neither makes it easier nor more difficult than it was before. Conics are specially adapted for treatment by line co-ordinates; and they suffice to bring out the fact that to every property concerning points and lines there is a corresponding property concerning lines and points. "Not to be too radical, however," the author has "left the balance of power with point co-ordinates."

After an introductory chapter the author explains what is meant by the co-ordinates of a point and the co-ordinates of a line; and shows that the condition that the point x, y and the line ξ, η are incident to one another is expressed by the equation $\xi x+\eta y+1=0$. This equation, then, is very important since it expresses symmetrically the condition that the line ξ, η passes through the point x, y, and that the point x, y lies on the line ξ, η. From this it is an easy step in the third chapter to show that the line ξ, η has $\xi x+\eta y+1=0$ for its equation in

point co-ordinates, and the point x, y has $x\xi+y\eta+1=0$ for its equation in line co-ordinates.

We cannot refer in detail to the other chapters; it suffices to say that the average student can easily master their contents (again excepting the last chapter), and that after doing so he will have a vivid idea of the elementary properties of conics, including those of diameters, tangents, asymptotes, poles and polars, and the circular points; and will have learnt to change the axes with ease, and to trace the conic from its equation. The last chapter deals with miscellaneous examples; these should certainly not be taken in the lump. Geometrical methods are used instead of analytical whenever they have the advantage in clearness and simplicity.

Homogeneous co-ordinates are not referred to, being reserved, we suppose, for Part II. Two subjects are, however, deliberately omitted, which would seem to have their natural place in Part I. The first of these is oblique axes, which we do not regard as a serious omission, because they are of use only in application to conics, and are not worth the time that has to be given to them. But we think that polar co-ordinates deserve more consideration than is given to them, especially in regard to the fact that the position of a point is expressed by them in more than one way, and consequently that one and the same curve may have two different equations. The tyro thinks that $S+\lambda S'$ contains all the points common to S and S'; and does not easily find out for himself that this is only true when S and S' are algebraic polynomials. One other criticism we may make. In solving some questions more parameters are introduced than are necessary. To obtain the equation to a line or curve it seems obvious that the best method, if feasible, is to use the co-ordinates of any point on the line required as the only parameters to solve the question with; for then only one equation has to be found, and this is the equation of the line. The book does not seem to us to adopt this method as consistently as it might do. It is much the same thing that we come across in Algebra, viz., that a problem is generally most easily solved by introducing as few unknowns as possible.

Plane Geometry for Secondary Schools. By CHARLES DAVISON, Sc.D., and C. H. RICHARDS, M.A. (Cambridge University Press.)

The authors tell us that the work was planned in 1903, but its execution was deferred until the experience of some years should indicate the merits and defects of the new methods of teaching geometry. We turned therefore to the book with pleasant anticipations, especially as we were in complete accord with all the opinions expressed in the preface. We must confess, however, to disappointment. Most of the features of the book are excellent, the introduction is masterly; but the authors have not thrown off, or have deliberately resumed, some of their old shackles. They have gone back to the idea of treating elementary geometry as a subject apart, instead of bringing it into relation with arithmetic and mensuration. Apart from this, the book does not go far enough, especially for pupils who will not proceed to a more advanced work. It is some little time since we read the book through, but in glancing at it again we can find no mention of centres of similitude, coaxal systems of circles, maximum and minimum, poles and polars, and inversion.

What we regard as a very retrograde feature of the book is its treatment of areas. Although in the treatment of proportion all lines are regarded as if they were commensurable, there seems to be a reluctance to do the same in the still earlier treatment of area. It is nowhere proved that the area of a rectangle is measured by the product of its two sides; although the proper place to prove this is surely in an elementary geometry; following it with the corresponding formulae for parallelograms and triangles. The whole subject of area has been made needlessly obscure, when it might have been made as plain as a pikestaff to the densest intellect. So again the idea of ratio should be led up to by an explanation of multiple, sub-multiple, and fraction. Instead of this we are told that "it will be assumed that the student is acquainted with the algebraic theory of proportion as applied to abstract numbers." There is no harm in repetition; and for a student to read an explanation of the same thing in two different books is extremely helpful. But there is considerable harm in a statement of this kind, when the student and his guide do not know where to look for a satisfactory algebraic theory.

The proof of Prop. 10, p. 37, is insufficient, since it would equally apply to the case of any two triangles with two pairs of equal sides, and one pair of equal angles opposite one pair of the equal sides. F. S. MACAULAY.

Encyclopädie der Elementar-mathematik. Dritter Band. Angewandte Elementar-mathematik. H. WEBER, J. WELLSTEIN, and R. H. WEBER. Pp. xiii, 666. Price, 14 m. 1907. (Teubner, Leipzig.)

This third volume of the Encyclopedia of Elementary Mathematics is devoted to applied mathematics. As the authors explain in the preface, this volume is not encyclopedic in the ordinary sense, for it makes no pretence to a complete treatment of the subjects dealt with. It is intended rather as a careful exposition of the principles of those branches of applied mathematics from which school courses are compiled.

The interest of the book to an English teacher lies chiefly in the selection and relative proportion of the topics dealt with, namely, Vector Geometry, Analytical Statics and Dynamics, Electric and Magnetic Lines of Force, Geometric Maxima and Minima, Probability and Theory of Errors, Descriptive Geometry, Graphic Statics.

Is the fact that 140 pages are devoted to electricity a sign of the times?

Vorlesungen über darstellende Geometrie. (I.) Die Darstellungsmethoden. By DR. GINO LORIA. Translated from the Italian MS. by FRITZ SCHÜTTE. Pp. xi, 219. Price, 6 m. 80. 1907. (Teubner, Leipzig.)

The subject of descriptive geometry, founded on the labours of Monge, has received little attention in recent years from English mathematicians; who indeed, using 'descriptive' as antithetic to 'metrical,' are sometimes puzzled for a moment by the phrase 'descriptive geometry' universally applied by students to the arts of representing solids by plane figures.

Thus didactic work in this country on geometry has dealt almost exclusively with plane geometry, while, on the other hand, 'practical' manuals of descriptive geometry, neglecting to use the powerful methods of modern projective geometry, and failing to treat the subject as a coherent branch of mathematics, have been apt to consist of sets of detached problems, the beads without the string.

The increased weight (exemplified by the new regulations for the Tripos) which we are being led to attach to solid geometry will in course of time bring about a remedy, and we shall be prepared to give the answer implied in Professor Klein's question (quoted by Professor Loria) whether correct drawing is not just as important as correct computing.

The present volume, which is based on a course of lectures at the University of Genoa, deals with the line and plane, and contains five books or sections on Monge's method of plan and elevation, conical projection or perspective, contoured plans, axonometric projections, and the applications of perspective methods to the reconstruction of objects from photographs.

This last section deals with two fundamental problems which have recently received much attention from military topographers on the Continent, namely, (1) Given two photographs of an object from different points of view to construct a diagram of the object from a third point of view. (2) Given 'n' photographs of an object from unknown points of view to find out as much as possible of its shape and position, and in particular, whether the object can be reconstructed (save for scale) from the data.

Professor Loria deals with methods rather than with detailed applications. His work is notable for its unity and scientific arrangement, for the generality of the treatment, and for the care taken to consider the practicability of the constructions given and to indicate methods of checking the results.

The work was translated from the Italian MS. by Herr Schütte, who also drew the diagrams. C. S. JACKSON.

QUERIES.

(24) What is the derivation of the term "mantissa"? LOG.

(25) When was the term "sine" first used? Who is responsible for "haversine"? Is it merely "half-versine"? PECTUS.

(26) Can the Gamma function be defined for imaginary values of the variable? Γ.

(27) If $\sqrt{n}$ be expanded as a continued fraction, is there any limit to the number of the quotients? L.

(28) Given the three vertices of the three equilateral triangles described on the sides of a triangle either inwards or outwards, or some inwards and others outwards, find the vertices of the original triangle. REVERS.

(29) Can a triangle be constructed if we know the feet of its internal bisectors, or of its three external bisectors, or of two internal and one external, or of two external and one internal? BIS.

(30) What is the origin of the sign % to denote per cent.? MOSES.

(31) Does any work on Geometry deal at length with constructions that may be made with the rule alone? S. EDGE.

(32) Have the sum of the squares and the sum of the cubes of the coefficients of $(1+x)^n$ ever been found? Σ.

(33) Required a proof of Euc. I. 8, which does not mention angles. U^4.

(34) Required a solution of the following:

Equal forces act at any point P on the nine-points circle of a triangle, and their directions pass through the middle points of the sides. Prove that their resultant passes through the point of contact of the nine-points circle with the inscribed circle, or, with one of the escribed circles.

V. RAMASWAMI AIYAR.

ANSWERS TO QUERIES.

[23, p. 131.] A formula for the approximation of $n!$ when n is very great is that of Stirling (*v.* Chrystal's Algebra, Vol. II., p. 344).

$$\underline{|n} = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \exp.\left\{\frac{1}{12n}+\theta\right\},$$

or in another form $$= \sqrt{2\pi n}\, n^n e^{-n}\left(1+\frac{1}{12n}+\frac{1}{288n^2}+\dots\right),$$

or $$\operatorname*{Lt}_{n=\infty} \underline{|n} = \sqrt{2\pi n}\,.\,n^n\,.\,e^{-n}.$$ H. G. MAYO.

$\dfrac{\underline{|n}}{n^n e^{-n}\sqrt{2n\pi}}$ is expanded in a series in $\dfrac{1}{n}$, in Boole's *Finite Differences*, *v.* p. 93, 3rd Edition, 1880. W. P. WORKMAN.

[19, p. 131.] The highest primes known are:

$2^{61}-1$ discovered by Seelhoff, confirmed by Prof. Cole (U.S.).

$5\cdot 2^{75}+1$ a divisor of $2^{273}+1$ discovered by Dr. Morehead.

$2^{128}+1$ discovered by the late Ed. Lucas.

As to "frequency" of primes, see the Introduction to Jas. Glaisher's *Factor-Table for the 6th Million*, 1883, passim.

For a bibliography of the subject up to 1879, see the Introduction to Jas. Glaisher's *Factor-Table for the 4th Million*, pp. 36-38, London, 1879. Other papers have appeared since 1879. ALLAN CUNNINGHAM, Lt.-Col. R.E.

The largest prime number at present known is still, I believe,

2,305,843,009,213,693,951.

The best account of the distribution of primes is contained in the prefaces of Glaisher's *Factor-Tables for the 4th and 6th Millions*, particularly the latter. W. P. WORKMAN.

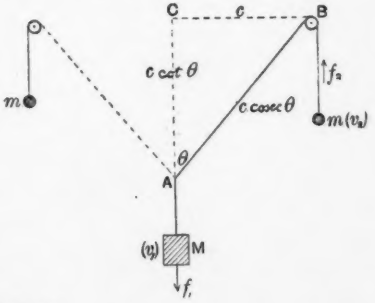
MATHEMATICAL NOTES.

249. [I. 2. b.] Mr. Workman in the July *Gazette* gives a table of the number of places in the periods of radix-fractions in scales other than the denary. If we used some of these scales instead of the denary, an obscure rule concerning such fractions would be brought into much more prominent notice than at present. The rule I refer to is that which states that $\frac{1}{n^q}$ (n prime and q any integer) has a period of $p.n^{q-1}$ figures, where p is the number of figures in the period of $\frac{1}{n}$, as for example $\frac{1}{7^3}$ has a period of 6×7^2 figures. The only known exceptions in the denary scale are $\frac{1}{3^2}$ and $\frac{1}{487^2}$, as quoted by Prof. Glaisher from Desmarest. But in the scale of radix 19, $\frac{1}{3^2}$, $\frac{1}{7^2}$ $\left(\text{and } \frac{1}{7^3}\right)$, $\frac{1}{13^2}$, $\frac{1}{43^2}$ have the same number of figures in their periods as $\frac{1}{3}$, $\frac{1}{7}$, $\frac{1}{13}$ and $\frac{1}{43}$ respectively. There are numerous other cases, *e.g.* in scale of 53, $\frac{1}{2^2}$, $\frac{1}{3^2}$ $\left(\text{and } \frac{1}{3^3}\right)$, $\frac{1}{47^2}$ and $\frac{1}{59^2}$ have periods of same length as $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{47}$ and $\frac{1}{59}$. This would seem to suggest that there are other exceptions in the denary scale besides the two that are known.

Is it certain that the rule as enunciated above is true in another respect? Is it impossible for $\frac{1}{n^3}$ to have a period of same length as $\frac{1}{n^2}$ without $\frac{1}{n^2}$ having a period of same length as $\frac{1}{n}$, *e.g.* though $\frac{1}{7^2}$ and $\frac{1}{7^3}$ obey the rule in the denary scale, does it of necessity follow that $\frac{1}{7^4}$ cannot recur in a period of same length as the period of $\frac{1}{7^3}$?

Kimberley, S.A. C. C. WILES.

250. [R. 7. a.] The solution of 581, Vol. IV., p. 62, is correct, as the question stands, only for initial motion from rest at an angle θ. The principle is of course correct, but must be applied very warily in this case.



The acceleration of A

=acceleration of A relative to the point of string in contact with B
+acceleration of this point of the string relative to the fixed point B. (See figure.)

That is, the principle crudely stated only applies to a string moving in any manner, when the *motion of each end of the string is tangential to the curve of the string at that end.*

Hence $f_1 \cos\theta$ = acceleration of A along BA

$$= -r\omega^2 + f_2,$$

and $$\omega = \frac{v_1}{c}\sin^2\theta, \quad r = c\operatorname{cosec}\theta;$$

$$\therefore\ f_2 = f_1\cos\theta + \frac{v_1^2}{c}\sin^3\theta. \qquad \ldots\ldots(1)$$

This equation is more easily obtained by the methods of the calculus; for neglecting sign for the moment,

$$v_1 = \frac{d}{dt}.c\cot\theta = -c\operatorname{cosec}^2\theta.\dot{\theta},$$

$$v_2 = \frac{d}{dt}.c\operatorname{cosec}\theta = -c\operatorname{cosec}\theta.\cot\theta.\dot{\theta};$$

$$\therefore\ v_2 = v_1\cos\theta; \qquad \ldots\ldots(2)$$

$$\therefore\ f_2 = f_1\cos\theta - v_1\sin\theta.\dot{\theta}$$

$$= f_1\cos\theta + \frac{v_1^2}{c}\sin^3\theta.$$

Now as A originally coincided with C, the *vis-viva* equation is

$$Mv_1^2 + 2mv_2^2 = \{M.\cot\theta - 2m(\operatorname{cosec}\theta - 1)\}cg;$$

$$\therefore\ v_1^2[M + 2m\cos^2\theta] = \{M.\cot\theta - 2m(\operatorname{cosec}\theta - 1)\}cg. \qquad \ldots\ldots(3)$$

Taking the static equation, we have

$$T - mg = mf_2, \qquad \ldots\ldots(4)$$

$$Mg - 2T\cos\theta = Mf_1; \qquad \ldots\ldots(5)$$

$$\therefore\quad g(M - 2m\cos\theta) = Mf_1 + 2mf_2\cos\theta$$

$$= Mf_1 + 2m\cos^2\theta f_1 + 2m\frac{v_1^2}{c}\cos\theta\sin^3\theta;$$

$$\therefore\ f_1'(M + 2m\cos^2\theta) = (M - 2m\cos\theta)g$$

$$- 2m\cos\theta\sin^3\theta\left\{\frac{M\cot\theta - 2m(\operatorname{cosec}\theta - 1)}{M + 2m\cos^2\theta}\right\}g;$$

$$\therefore\ f_1 = \frac{M - 2m\cos\theta}{M + 2m\cos^2\theta}g - \frac{2m\cos\theta\sin^2\theta[M\cos\theta - 2m(1 - \sin\theta)]}{(M + 2m\cos^2\theta)^2}g.$$

But in this case equation (2) does not hold good for differentiating, since it has been obtained by dividing by the zero $\dot{\theta}$. It, however, comes to the same thing in the end.

Hence for initial motion at angle θ,

$$f_1 = \frac{d^2}{dt^2}.c\cot\theta = +2c\operatorname{cosec}^2\theta\cot\theta.\dot{\theta}^2 - c\operatorname{cosec}^2\theta.\ddot{\theta},$$

$$f_2 = \frac{d^2}{dt^2}.c\operatorname{cosec}\theta = +c\operatorname{cosec}\theta\cot^2\theta.\dot{\theta}^2 + c\operatorname{cosec}^3\theta.\dot{\theta}^2 - c\operatorname{cosec}\theta.\cot\theta.\ddot{\theta},$$

putting $\dot{\theta} = 0$, $\qquad f_2 = f_1\cos\theta,$

as might have been expected from (1), obtained by elementary considerations.

J. M. CHILD.

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